MATH 20D: Differential Equations Spring 2023
Homework 8

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Remember to list the sources you used when completing the assignment.
Below *NSS* is used to reference the text *Fundamentals of Differential Equations (9th edition) by Nagle, Saff, Snider*

Question (1). *Let* $A = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, *and* $C = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$. *Calculate*

(a) AB	***(b)*** AC	***(c)*** $A(B+C)$	***(d)*** A^{-1}
(e) B^{-1}	***(e)*** $B^{-1}A^{-1}$	***(f)*** $(AB)^{-1}$	***(h)*** $\det(AC)$.

Is it true that $AC = CA$? *Do we have* $\det(AC) = \det(CA)$? *Is it always the case that* $\det(XY) = \det(YX)$ *for a pair of 2-by-2 matrices* X *and* Y? *Do we have* $(XY)^{-1} = Y^{-1}X^{-1}$? *Can* XY *be invertible if either* $\det(X) = 0$ *or* $\det(Y) = 0$?

Question (2). (NSS 9.1.1,2, & 5, NSS 9.4.1, & 2, and NSS 9.3.17,18, & 35)

(a) Express the each of the systems below as matrix equations in normal form

(i) $\begin{aligned} x' &= 7x + 2y, \\ y' &= 3x - 2y \end{aligned}$ ***(ii)*** $\begin{aligned} x' &= y, \\ y' &= -x \end{aligned}$ ***(iii)*** $\begin{aligned} x' &= \sin(t)x + e^t y, \\ y' &= \cos(t)x + (a + bt^3)y \end{aligned}$

(iv) $\begin{aligned} x'(t) &= 3x(t) - y(t) + t^2, \\ y'(t) &= -x(t) + 2y(t) + e^t \end{aligned}$ ***(v)*** $\begin{aligned} r'(t) &= 2r(t) + \sin(t), \\ \theta'(t) &= r(t) - \theta(t) + 1. \end{aligned}$

(b) For each of the matrices $X(t)$ *below calculate the matrix* $X^{-1}(t)$

(i) $X(t) = \begin{pmatrix} e^t & e^{4t} \\ e^t & 3e^{4t} \end{pmatrix}$ ***(ii)*** $X(t) = \begin{pmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{pmatrix}$.

(c) Verify that the function $x(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ *satisfies the matrix differential equation*

$$x'(t) = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x(t).$$

Question (3). (NSS 9.1.7,8, & 9 and NSS 9.4.5, & 6) *Express each of the second order differential equations below as a first order system in normal form*

(a) $3y'' + 2y' + 4y = 0$	***(b)*** $(1-t^2)y'' - 2ty' + 2y = 0$	***(c)*** $y'' - ty = 0$
(d) $y'' - ty = 0$	***(e)*** $y''(t) - 3y'(t) - 10y(t) = \sin(t)$	***(f)*** $x''(t) + x(t) = t^2$

Question (4)**.** (NSS 9.4.13,14,15, & 16) *Determine whether the given vector valued functions are linearly dependent or linearly independent.*

$$\textbf{\textit{(a)}}\quad \begin{pmatrix} t \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad\qquad \textbf{\textit{(b)}}\quad \begin{pmatrix} te^{-t} \\ e^{-t} \end{pmatrix}, \qquad \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

$$\textbf{\textit{(c)}}\quad e^t \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad e^t \begin{pmatrix} -3 \\ -15 \end{pmatrix} \qquad\qquad \textbf{\textit{(d)}}\quad \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \quad \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$$

Question (5)**.** *(a) Verify that the vector valued functions*

$$x_1(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix} \quad \textit{and} \quad x_2(t) = \begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix}$$

give a fundamental solution set to the differential equation

$$x'(t) = Ax(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x(t).$$

(b) Write down a general solution to the equation $x'(t) = Ax(t)$.

Question (6)**.** (NSS 9.4.28 & 30)
(a) Verify that

$$X(t) = \begin{pmatrix} e^{-t} & e^{5t} \\ -e^{-t} & e^{5t} \end{pmatrix}$$

is a fundamental matrix for the linear system $x'(t) = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} x(t)$.

(b) Let A be a 2-by-2 matrix and suppose $X(t)$ is the fundamental matrix for the system $x'(t) = Ax(t)$. Show that if x_0 is a 2-by-1 constant matrix then

$$x(t) = X(t)X^{-1}(0)x_0$$

solves the initial value problem

$$x'(t) = Ax(t), \qquad x(0) = x_0$$

(c) Solve the linear system in (a) subject to the initial condition $x(0) = \begin{pmatrix} 3 & -1 \end{pmatrix}^T$.

Question (7)**.** (NSS 9.5.11 & 12 and NSS 9.5.31 & 32)
(a) For each of the matrices A below find a general solution to the system $x'(t) = Ax(t)$

$$\textbf{\textit{(i)}}\quad A = \begin{pmatrix} -1 & 3/4 \\ -5 & 3 \end{pmatrix} \qquad\qquad \textbf{\textit{(ii)}}\quad A = \begin{pmatrix} 1 & 3 \\ 12 & 1 \end{pmatrix}$$

(b) Solve each of the initial value problems below

$$\textbf{\textit{(i)}}\quad x'(t) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} x(t), \quad x(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \textbf{\textit{(ii)}}\quad x'(t) = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} x(t), \quad x(0) = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$$

Question (8)**.** (NSS 9.6.1 & 2, NSS 9.6.5 & 4, and NSS 9.6.13)
(a) Find a general solution to the system $x'(t) = Ax(t)$ for each of the matrices A below

$$\textbf{\textit{(i)}}\quad A = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix} \qquad \textbf{\textit{(ii)}}\quad A = \begin{pmatrix} -2 & -5 \\ 1 & 2 \end{pmatrix}.$$

(b) Find a fundamental matrix for the system $x'(t) = Ax(t)$ for each of the matrices A below

$$\textbf{\textit{(i)}}\quad A = \begin{pmatrix} -1 & -2 \\ 8 & -1 \end{pmatrix} \qquad \textbf{\textit{(ii)}}\quad A = \begin{pmatrix} -2 & -2 \\ 4 & 2 \end{pmatrix}.$$

(c) In part (i)-(iv) below, solve the system $x'(t) = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} x(t)$ for the given initial condition

$$\textbf{\textit{(i)}}\quad x(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \textbf{\textit{(ii)}}\quad x(\pi) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \textbf{\textit{(iii)}}\quad x(-2\pi) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \textbf{\textit{(iv)}}\quad x(\pi/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Question (9)**.** (NSS 9.5.45) *Two tanks, each holding 50 L of liquid are interconnected by pipes with liquid flowing from tank A into tank B at a rate of 4 L/min and from tank B into tank A at a rate of 1 L/min. The liquid inside each tank is kept well stirred. Pure water flows into tank A at a rate of 3 L/min, and the solution flows out of tank B at 3 L/min. If, initially, tank A contains 2.5 kg of salt and tank B contains no salt (only water), determine the mass of salt in each tank at time $t \geq 0$.*